

On the Expectations of Maxima of Sets of Independent Random Variables

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Abstract

Let X^1, \dots, X^k and Y^1, \dots, Y^m be jointly independent copies of random variables X and Y , respectively. For a fixed total number n of random variables, we aim at maximising $M(k, m) := \mathbb{E} \max\{X^1, \dots, X^k, Y^1, \dots, Y^m\}$ in $k = n - m \geq 0$, which corresponds to maximising the expected lifetime of an n -component parallel system whose components can be chosen from two different types. We show that the lattice $\{M(k, m): k, m \geq 0\}$ is concave, give sufficient conditions on X and Y for $M(n, 0)$ to be always or ultimately maximal and derive a bound on the number of sign changes in the sequence $M(n, 0) - M(0, n)$, $n \geq 1$. The results are applied to a mixed population of Bienayme-Galton-Watson processes, with the objective to derive the optimal initial composition to maximise the expected time to extinction.

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1 Introduction

Consider a simple n -component parallel system (called an n -assembly in what follows), whose component lifetimes are independent non-negative random variables X_1, \dots, X_n with finite means. The n -assembly fails when all the n components fail, and so its expected lifetime is given by $\mathbb{E} \max\{X_1, X_2, \dots, X_n\}$, which is a standard measure of the system's reliability.

Such quantities and related characteristics of complex systems often appear in literature on reliability. For instance, comparisons and bounds for expected lifetimes of series and parallel systems are considered in [8], while assemblies whose

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components' lifetime distributions can depend on the amount of "investment" are discussed in [7]. Expected lifetimes conditional on survival up until time t are investigated in [1], while assemblies of dependent components with interchangeable lifetime distributions are analysed in [9]. More recently, upper and lower bounds for the expected lifetimes of n -assemblies of components with (possibly) non-identically distributed lifetimes in terms of the expected lifetimes of assemblies with identically distributed components' lifetimes were given in [6].

In the present study, we assume that, when building an n -assembly, one may choose components from d different types that have different lifetime distributions. Should one choose all n components of the same type (*unmixed* assembly) or should one mix? In general, the answer will depend not only on the distributions of the random lifetimes of the components, but also on n . In this paper we shall examine these relationships.

Questions of such kind arise naturally in reliability problems such as server failures, where information is duplicated on multiple hard disks. In conservation biology, the "Single Large or Several Small" problem provides another application. Here one should decide between the creation of a single large protected habitat or several small ones. This may be modelled by n -assemblies since the lifetimes of individuals in the species in different small habitats may be modelled by random variables X_i with distinct (due to differing environmental conditions) distributions. In contrast, a single large habitat with the same lifetime distribution for all individuals in the species corresponds to an n -assembly composed of identically distributed random variables. Furthermore, the question of whether mutation is inherently advantageous because it leads to diversity may also be analysed in a similar way.

In this paper, we first show that the expected lifetime $M(k_1, \dots, k_d)$ of an assembly consisting of k_i components of type $i = 1, \dots, d$, is a concave function of (k_1, \dots, k_d) . This implies that one can easily find the optimal n -assembly in the general case using the discrete steepest ascent method in linear time in dn . After that we focus on the case of two component types, with a view to maximising the life expectancy of the system, noting that generalisations to more than two types are straightforward and are omitted for the sake of simplicity. We denote the two random lifetimes by X and Y and their respective distribution functions by F and G , always assuming that $\mathbb{E}X + \mathbb{E}Y < \infty$, and assume that X_1, X_2, \dots are i.i.d. like X , and Y_1, Y_2, \dots are i.i.d. like Y , with all the X_i 's and Y_j 's being jointly independent. Our main object of study is

$$M(k, m) := \mathbb{E} \max\{X_1, \dots, X_k, Y_1, \dots, Y_m\} = \int_0^\infty (1 - F(s)^k G(s)^m) ds.$$

We classify the possible behaviour of these quantities, e.g. when it is advantageous to always choose one type and when it is the right choice eventually, giving sufficient conditions for these. We also give a simple bound for the number of sign changes of $M(n, 0) - M(0, n)$ and apply the aforementioned results to the case of Bienayme-

Galton-Watson processes, with a view to selecting an optimal initial composition in order to maximise the expected time to extinction of the entire population. But first we present a couple of simple examples that show some possible types of behaviour that can be displayed by $M(k, m)$.

Example 1. Suppose we can choose from two types of components: one with lifetimes $X \sim U[0, 1]$ (i.e. distributed uniformly on $[0, 1]$) and the other with deterministic lifetimes $Y \equiv \varepsilon = \text{const.}$ Clearly, if $\varepsilon \geq 1$ then one should always choose components of type Y . In general, we will refer to a situation where the optimal (with the longest expected lifetime) n -assembly is always composed of one particular type of component as the *dominant* case, and to that component type as the *dominant* type.

Now assume that $0 < \varepsilon < 1$. The choice is between selecting all n components of the first type or $n - 1$ components of the first type and one component of the second type (there is no point in taking more than one type two component to extend the expected lifetime as all such components will fail at precisely the same time). Integrating to obtain the corresponding expectations gives

$$M(n, 0) = \int_0^1 (1 - x^n) dx = 1 - \frac{1}{n+1},$$

$$M(n-1, 1) = \varepsilon + \int_\varepsilon^1 (1 - x^{n-1}) dx = 1 - \frac{1}{n} + \frac{\varepsilon^n}{n}.$$

We conclude that the assembly corresponding to $(n, 0)$ yields a higher expected lifetime than that for $(n-1, 1)$ for n such that $(n+1)^{1/n} < \varepsilon^{-1}$. If $\varepsilon < 1/2$, this will always be true, and then we have a dominant case. Since $\varepsilon < 1$ and $(n+1)^{1/n} \rightarrow 1$ as $n \rightarrow \infty$, we see that, regardless of ε , $M(n, 0) > M(n-1, 1)$ holds for all large enough n . In general, if unmixed assemblies $\{X_1, \dots, X_n\}$ are advantageous for all large enough n , we say that X is *ultimately dominant*. We will see (in Theorem 6) that this behaviour takes place whenever one distribution function ultimately dominates the other.

Since $(n+1)^{1/n}$ is a decreasing function of $n \geq 1$, we see that the values of n for which $(n+1)^{1/n} > \varepsilon^{-1}$, i.e. for which $(n-1, 1)$ is preferable to $(n, 0)$, form either an empty set or a finite sequence of successive integers. In the latter situation, since the optimal n -assemblies are composed of more than one component type, we say that we have a *non-dominant case* with the number of *dominance changes* (i.e. sign changes of $M(n, 0) - M(0, n)$) equal to 1.

We will see that this number admits a simple upper bound in terms of the distribution functions of X and Y (Theorem 7), but can actually be infinite, as the following example suggested by A. Sudbury shows.

Example 2. Let X and Y have distributions $\mathbb{P}(X = 2i) = p_i$ and $\mathbb{P}(Y = 2i+1) = q_i$, $i = 0, 1, 2, \dots$, such that

$$P_j := p_j + p_{j+1} + \dots = q_{j+1} + q_{j+2} + \dots = 2^{-2^j}, \quad j \geq 2.$$

Set $\mathcal{A}(k, m) := \max\{X_1, \dots, X_k, Y_1, \dots, Y_m\}$, so that $M(k, m) = \mathbb{E}\mathcal{A}(k, m)$.

We will show that there exist two increasing infinite sequences $\{k_i\}$ and $\{l_i\}$ such that $M(k_i, 0) > M(0, k_i)$ and $M(l_i, 0) < M(0, l_i)$ for all sufficiently large i .

Indeed, taking $k_i := \lceil 2^{2^i} i^{-2} \rceil$, where $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$, we see that

$$\mathbb{P}(\mathcal{A}(k_i, 0) < 2i) = (1 - P_i)^{k_i} = (1 - 2^{-2^i})^{2^{2^i} i^{-2}} = \exp\{-(1 + o(1))i^{-2}\} \rightarrow 1,$$

$$\mathbb{P}(\mathcal{A}(k_i, 0) < 2(i-1)) = (1 - 2^{-2^{i-1}})^{2^{2^i} i^{-2}} = \exp\{-(1 + o(1))2^{2^{i-1}} i^{-2}\} \rightarrow 0$$

as $i \rightarrow \infty$, so that $M(k_i, 0) \sim 2(i-1)$ (here \sim means asymptotic equivalence: $u_n \sim v_n$ as $n \rightarrow \infty$ iff $u_n/v_n \rightarrow 1$), whereas

$$\begin{aligned} \mathbb{P}(\mathcal{A}(0, k_i) \geq 2i+1) &= 1 - (1 - P_{i-1})^{k_i} \\ &= 1 - (1 - 2^{-2^{i-1}})^{2^{2^i} i^{-2}} = 1 - \exp\{-(1 + o(i))2^{2^{i-1}} i^{-2}\} \rightarrow 1. \end{aligned}$$

Thus $M(k_i, 0) < M(0, k_i)$ for all sufficiently large i .

Now let $l_i := 2^{2^{i+1}}(i+1)^{-2}$. Then it can be shown in a similar fashion that $M(l_i, 0) > M(0, l_i)$ for all sufficiently large i , so that we have infinitely many dominance changes in this example.

2 Concavity of the expected lifetimes and dominance

When maximizing the function $M(k_1, \dots, k_d)$, it is helpful to know that it is concave. In particular, this guarantees that, moving in the direction of the steepest ascent one can find the optimal composition in linear time in dn .

Theorem 3. *Let X_1, \dots, X_d be random lifetimes with corresponding distribution functions F_1, \dots, F_d . Then the function*

$$M(\mathbf{x}) := \int_0^\infty (1 - F_1(s)^{x_1} \dots F_d(s)^{x_d}) ds, \quad \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}_+^d,$$

is concave.

Clearly, the function $M(\mathbf{x})$ coincides with the corresponding systems' expected lifetimes at the integer lattice points.

Proof. We will begin with the obvious observation that, for any $\mathbf{x} \in \mathbb{R}_+^d$,

$$M(\mathbf{x}) \leq M(\lceil x_1 \rceil, \dots, \lceil x_d \rceil) \leq \sum_{j=1}^d \lceil x_j \rceil \mathbb{E}X_j < \infty.$$

Fix arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^d$ and let $F(s) := F_1^{x_1}(s) \cdots F_d^{x_d}(s)$, $G(s) := F_1^{y_1}(s) \cdots F_d^{y_d}(s)$. It suffices to show that the univariate function

$$M_1(z) := M(z\mathbf{x} + (1-z)\mathbf{y}) = \int_0^\infty (1 - F(s)^z G(s)^{1-z}) ds, \quad z \in [0, 1],$$

is concave. To justify differentiation inside the integral, we first fix an arbitrarily small $\delta > 0$ and show that the integral of the integrand's derivative converges absolutely and uniformly in $z \in (\delta, 1 - \delta)$. We have

$$\int_0^\infty \left| \frac{\partial}{\partial z} (1 - F(s)^z G(s)^{1-z}) \right| ds = \int_0^\infty |\ln F(s) - \ln G(s)| F(s)^z G(s)^{1-z} ds,$$

where the integrand is dominated by the function

$$|\ln F(s)| F(s)^\delta + |\ln G(s)| G(s)^\delta$$

(using the natural convention $0 \cdot \ln 0 = 0$), which is clearly bounded on $(0, \infty)$ and so is integrable there since $|\ln F(s)| F(s)^\delta \sim 1 - F(s)$ as $s \rightarrow \infty$.

In a similar way one can verify that M_1 is twice differentiable and

$$\begin{aligned} M_1''(z) &= \int_0^\infty \frac{\partial^2}{\partial z^2} (1 - F(s)^z G(s)^{1-z}) ds \\ &= - \int_0^\infty (\ln F(s) - \ln G(s))^2 F(s)^z G(s)^{1-z} ds < 0 \end{aligned}$$

unless $F(s) \equiv G(s)$. This shows that M_1 and hence M are concave. \square

Now we turn our attention to the case when one type is always dominant, i.e. one has $M(n, 0) \geq M(k, n - k)$ for any $n \in \mathbb{N}$, $0 \leq k \leq n$.

It is obvious that if

$$F(s) \leq G(s) \quad \text{for all } s \in \mathbb{R}_+, \quad (1)$$

then we have a dominant case. One can weaken (1) by requiring that one distribution always has a heavier integrated tail than the other.

Theorem 4. (Theorem 7.6 of [3]) *Let X and Y be random lifetimes with df's F and G respectively. If*

$$\int_s^\infty (F(t) - G(t)) dt < 0 \quad \text{for all } s \in \mathbb{R}_+, \quad (2)$$

then $M(n, 0) > M(k, n - k)$ for all $n \in \mathbb{N}$ and $k \in [0, n]$.

Recall that (2) defines the so-called increasing convex ordering on the set of distributions on \mathbb{R}_+ (see e.g. p.11 in [11]), whereas (1) is the standard stochastic ordering.

The next example shows that the above condition can be necessary in some special cases.

Example 5. Let X be uniformly distributed on $[0, 1]$ and Y uniformly distributed on $[a, a + \varepsilon]$ with $0 < a < a + \varepsilon < 1$. Then clearly

$$\begin{aligned} M(n, 0) - M(0, n) &= \int_0^1 (1 - x^n) dx - \left[a + \int_a^{a+\varepsilon} \left(1 - \left(\frac{x-a}{\varepsilon} \right)^n \right) dx \right] \\ &= (1 - \varepsilon) \left(1 - \frac{1}{n+1} \right) - a. \end{aligned}$$

Hence $M(n, 0) \geq M(0, n)$ iff $a/(1 - \varepsilon) \leq 1 - (n+1)^{-1}$. This is the case for all n (and then the first type is dominant) iff $a/(1 - \varepsilon) \leq 1/2$, and so this condition is necessary for the first type to dominate. On the other hand, by Theorem 4 for the X to be dominant it suffices that (2) holds, which is equivalent to the condition that the area of the upper of the two triangles formed by the graphs of the functions F and G is no less than the area of the lower one. Since the two triangles are similar, that means that the abscissa of the point of the intersection of the two graphs does not exceed $1/2$, i.e. $a/(1 - \varepsilon) \leq 1/2$. Thus we see that condition (2) which is sufficient for dominance of X is equivalent to the condition that was shown to be necessary for the dominance.

There is also a simple sufficient condition which ensures that one type will eventually be dominant.

Theorem 6. *Let X and Y be random lifetimes with the corresponding distribution functions F and G , and let there exist an $s_0 \in \mathbb{R}_+$ such that $F(s) > G(s)$ for all $s \geq s_0$. Then there exists an $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ and $1 \leq k \leq n$, $M(k, n - k) < M(0, n)$, i.e. Y is ultimately dominant.*

Proof. We will prove the result in the continuous case, an extension to the general case being obvious.

Let $s_0 = \inf\{s : F(s) > G(s), t > s\}$ and $F(s_0) = G(s_0) =: e^{-a}$. Then

$$\begin{aligned} M(k, n - k) - M(n, 0) &= \int_0^\infty (F(s)^k G(s)^{n-k} - G(s)^n) ds \\ &= \int_0^\infty G(s)^n \left(\left(\frac{F(s)}{G(s)} \right)^k - 1 \right) ds \\ &= \int_0^{s_0} + \int_{s_0}^\infty =: I_1 + I_2. \end{aligned}$$

Clearly, $|I_1| \leq s_0 e^{-an}$. Now set $H(s) := \ln F(s) - \ln G(s) \geq 0$ for $s > s_0$ and choose $s_1 > s_0$ such that $G(s_1) =: e^{-b} > e^{-a}$. Then

$$I_2 = \int_{s_0}^{\infty} G(s)^n (e^{kH(s)} - 1) ds \geq \int_{s_1}^{\infty} G(s)^n (e^{kH(s)} - 1) ds \geq k e^{-bn} \int_{s_1}^{\infty} H(s) ds.$$

Since $a - b > 0$, one has $\int_{s_1}^{\infty} H(s) ds > s_0 e^{-(a-b)n}$ for all large enough n , which clearly implies that, for all $k \geq 1$, $n \geq n_0$,

$$I_2 \geq k e^{-bn} \int_{s_1}^{\infty} H(s) ds > s_0 e^{-an} \geq |I_1|,$$

as required. \square

Using Theorem 6, one can easily find examples of non-dominant cases. It suffices to have $\mathbb{E}X > \mathbb{E}Y$ and $F(s) > G(s)$ for $s > s_0 \in \mathbb{R}_+$, so that $M(1, 0) - M(0, 1) > 0$, but $M(n^*, 0) - M(0, n^*) < 0$ for some $n^* \in \mathbb{N}$ (in other words, at least one sign change of $M(n, 0) - M(0, n)$ occur between $n = 1$ and $n = n^*$). In such situations, there typically exist n and $k^* \in (0, n)$ such that $M(k^*, n - k^*) \geq M(k, n - k)$ for all $0 \leq k \leq n$, i.e. the mixed assembly $(k^*, n - k^*)$ is optimal.

3 The number of sign changes of $M(n, 0) - M(0, n)$

One can obtain a useful upper bound for the number of sign changes in the sequence $M(n, 0) - M(0, n)$, $n \geq 1$, by generalizing the Descartes rule of signs. Clearly, this number does not exceed the number of zeros of the function

$$\psi(x) := \int_0^{\infty} (1 - F(s)^x) ds - \int_0^{\infty} (1 - G(s)^x) ds, \quad x > 0.$$

In order to find a bound for this number, we will employ a generalisation of the Descartes rule of signs [5].

One says that an ordered system $\{u_1, u_2, \dots\}$ of functions defined on $I \subseteq \mathbb{R}$ satisfies the Descartes rule of signs on I if the number of zeros (with multiplicities) of a linear combination $\sum a_j u_j(x)$ of these functions is less than or equal to the number of variations of strict sign in the sequence $\{a_1, a_2, \dots\}$ of the coefficients in the combination.

A system of functions $\{u_1, \dots, u_n\}$ defined on $I \subseteq \mathbb{R}$ is said to be strictly sign regular if, for any fixed $k \leq n$ and any $x_1 < x_2 < \dots < x_m$, $m \geq k$, with all $x_j \in I$, $j \leq m$, all minors of order k of the matrix $(u_j(x_i))_{i \leq m, j \leq n}$ have the same strict sign. For example, systems of the form $\{x^{\beta_1}, \dots, x^{\beta_n}\}$ and $\{e^{-\beta_1 x}, \dots, e^{-\beta_n x}\}$ with $0 < \beta_1 < \beta_2 < \dots < \beta_n$ are strictly sign regular on \mathbb{R} (see e.g. p.34 of [5]).

Proposition 2.6 of [5] asserts that a system of functions $\{u_1, \dots, u_n\}$ satisfies the Descartes rule of signs iff it is strictly sign regular.

Given two distribution functions F and G on \mathbb{R}_+ , let

$$\mu(A) := \int_A [F^{-1}(e^{-t}) - G^{-1}(e^{-t})] dt$$

be a signed measure on \mathbb{R}_+ . Let S^+ be a positive set and $S^- = \mathbb{R}_+ \setminus S^+$ a negative set from the Hahn-Jordan decomposition $\mu = \mu^+ - \mu^-$. We assume (for the moment) that S^+ has only finitely many connected components $S_1^+, \dots, S_{\nu_+}^+$ and that, without loss of generality, $0 \in S_1^+$. By $S_1^-, \dots, S_{\nu_-}^-$ we similarly denote the connected components of S^- , so that $x_1^+ < x_1^- < x_2^+ < x_2^- < \dots$ for any $x_i^\pm \in S_i^\pm$ (note that $0 \leq \nu_+ - \nu_- \leq 1$). Let $\nu = \nu^- + \nu^+$ and introduce sets Q_1, \dots, Q_ν by putting $Q_1 := S_1^+$, $Q_2 := S_1^-$, $Q_3 := S_2^+$, $Q_4 := S_2^-$ etc. The quantity ν may be called the number of intersections of F and G since ν^+ and ν^- are none other than the numbers of connected components of $\{s : F(s) \leq G(s)\}$ and $\{s : F(s) \geq G(s)\}$, respectively. When S^+ has infinite number of connected components, we set $\nu = \infty$. This was precisely the case in Example 2 from the Introduction.

Theorem 7. *Let X and Y be random lifetimes, F and G being their corresponding distribution functions. Then the number of sign changes of $M(n, 0) - M(0, n)$, $n \geq 1$, does not exceed $\nu - 1$.*

Proof. Noting that $X \stackrel{d}{=} F^{-1}(U)$ and $Y \stackrel{d}{=} G^{-1}(U)$, where $U \sim U[0, 1]$ and $H^{-1}(u) := \inf\{t : H(t) > u\}$ denotes the generalized inverse of the distribution function H , we can write

$$\psi(x) = \int_0^1 F^{-1}(s^{1/x}) ds - \int_0^1 G^{-1}(s^{1/x}) ds.$$

Letting $s^{1/x} = e^{-t}$, we see that $\psi(s) = x\psi_1(x)$, where

$$\begin{aligned} \psi_1(x) &= \int_0^\infty e^{-xt} \mu(dt) = \sum_{j=1}^\nu \int_{Q_j} e^{-xt} \mu(dt) \\ &= \sum_{j=1}^\nu (-1)^{j+1} \int_{Q_j} e^{-xt} |\mu(dt)| = \sum_{j=1}^\nu a_j \mathbb{E} e^{-xB_j} \end{aligned}$$

with $a_j = \mu(Q_j)$ (note that $\text{sign}(a_j) = (-1)^{j+1}$) and $B_j = H_j^{-1}(U_j)$, where

$$H_j(t) := \frac{1}{a_j} \int_0^t \mathbb{I}_{Q_j}(s) \mu(ds)$$

are distribution functions supported by the corresponding sets Q_j , $j = 1, \dots, \nu$, and \mathbb{I}_A denotes the indicator of A , while $U_j \sim U[0, 1]$ are independent random variables.

To show that $\{u_j(x)\} := \{\mathbb{E}e^{-x B_j}, j = 1, \dots, \nu\}$ is a strictly sign regular system of functions on \mathbb{R}_+ , first note that $\{v_j(x)\} := \{e^{-\beta_j x}\}$ is such a system provided that $0 < \beta_1 < \beta_2 < \dots < \beta_\nu$ (see p. 34 and Corollary 3.9 of [5]). From this it follows that, for a $k \leq \nu$, the sign of $\det(e^{-\beta_j x_i})_{i,j \leq k}$ is one and the same for any choice of $0 < \beta_1 < \dots < \beta_k$ and $0 < x_1 < \dots < x_k$.

Now observe that a k th order minor of $(u_j(x_i))_{j \leq \nu, i \leq m}$ admits an integral representation in terms of the corresponding minors of $(v_j(x_i))_{j \leq \nu, i \leq m}$. Indeed, consider without loss of generality the determinant of $(u_j(x_i))_{i,j \leq k}$ for $0 < x_1 < \dots < x_k$. It has the form

$$\begin{aligned} \sum_{\sigma} \text{sgn}(\sigma) \prod_{j=1}^k u_j(x_{\sigma(j)}) &= \sum_{\sigma} \text{sgn}(\sigma) \prod_{j=1}^k \mathbb{E} e^{-x_{\sigma(j)} B_j} = \mathbb{E} \sum_{\sigma} \text{sgn}(\sigma) \prod_{j=1}^k e^{-x_{\sigma(j)} B_j} \\ &= \int \dots \int \sum_{\sigma} \text{sgn}(\sigma) \left(\prod_{j=1}^k e^{-\beta_j x_{\sigma(j)}} \right) dH_1(\beta_1) \dots dH_k(\beta_k), \end{aligned}$$

where the summation is over all permutations σ of $\{1, \dots, k\}$, $\text{sgn}(\sigma)$ is the signature of σ and we used the independence of B_j . To complete our argument, it remains to make use of the above remark concerning the sign invariance of the determinants of the matrix $(e^{-\beta_j x_i})$ and recall that, by construction, we have $\beta_1 < \beta_2 < \dots < \beta_k$ in the integrand in the last displayed formula, due to $\beta_j \in Q_j$, $j \leq k$.

Thus we have shown that our $\{u_j(x)\}$ is a strictly sign regular system and so, by the Descartes rule, ψ_1 (and therefore ψ) can have at most $\nu - 1$ zeros in $(0, \infty)$. Thus $M(n, 0) - M(0, n)$ can have at most $\nu - 1$ sign changes, as required. \square

Note that it follows from Theorem 6 that if the (possibly infinite) sequence of points of intersection of the graphs of the distribution functions F and G is bounded, the number of dominance changes will necessarily be finite. However, if this sequence is unbounded (so that $\nu = \infty$), the number of dominance changes can be infinite as Example 2 in fact shows.

4 Mean time to extinction of mixed branching populations

In this section we consider an application of the above results to the optimal selection for populations of simple branching processes, where n -assemblies are composed of the times to extinction of the progenies of the individuals comprising the initial population.

Suppose we wish to establish a colony populated by two species, S_1 and S_2 , with the aim of maximising the colony's expected lifetime. Given a fixed initial total number n of individuals in the colony, we can choose k individuals of species S_1

and $n - k$ individuals of species S_2 . Next we assume that time is discrete and that the reproduction of individuals in the colony is governed by independent subcritical Bienayme-Galton-Watson (BGW) processes specific to the species.

Let $\xi \geq 0$ be an offspring random variable with a distribution $p_i = \mathbb{P}(\xi = i)$, $i \geq 0$, and $\xi_{n,i}$ with $n, i = 1, 2, \dots$ be independent copies of ξ . Further, for $r \in \mathbb{N}$, let $Z^{(r)}(0) := r$ and

$$Z^{(r)}(n) := \sum_{i=1}^{Z^{(r)}(n-1)} \xi_{n,i}, \quad n \geq 1.$$

Then $\{Z^{(r)}(n), n \geq 0\}$ is a BGW process starting with r individuals (for $r = 1$, we will simply write $\{Z(n)\}$).

Denote by $T^{(r)} := \inf\{n \geq 1 : Z^{(r)}(n) = 0\}$ the time to extinction of the process $\{Z^{(r)}(n)\}$ (we write T for $T^{(1)}$). Observe that $T^{(r)} = \max\{T^1, \dots, T^r\}$, where the T^j 's are independent copies of T . Let $f(\theta) := \sum_{i=0}^{\infty} p_i \theta^i$ be the probability generating function (pgf) of ξ and $f_n(\theta)$ the n th functional iterate of f . Then

$$\mathbb{P}(T \leq n) = \mathbb{P}(Z(n) = 0) = f_n(0), \quad n \geq 0, \quad (3)$$

and hence $\mathbb{P}(T^{(r)} \leq n) = f_n(0)^r$ and $\mathbb{E}T^{(r)} = \sum_{n=0}^{\infty} (1 - f_n(0)^r)$, where we put $f_0(0) = 0$.

Now denote by $\{Z_i(n)\}$ a BGW process describing the growth of a population of type S_i individuals (starting with a single individual at time 0), $i = 1, 2$, and by T_i the respective times to extinction. We will keep the notation f, f_n for the offspring pgf and its iterates for species S_1 and denote the respective pgf's for S_2 by g and g_n . Then the expected time to the extinction of our colony initially consisting of k individuals of species S_1 and $n - k$ individuals of species S_2 is given by

$$\begin{aligned} M(k, n - k) &= \mathbb{E} \max\{T_1^{(k)}, T_2^{(n-k)}\} \\ &= \mathbb{E} \max\{T_1^1, \dots, T_1^k, T_2^1, \dots, T_2^{n-k}\} = \sum_{i=0}^{\infty} (1 - f_i(0)^k g_i(0)^{n-k}), \end{aligned}$$

where the T_i^j are independent copies of T_i , $i = 1, 2$. The problem we want to consider is how to choose k that maximises $M(k, n - k)$ for a fixed n . First we obtain some preliminary facts.

Theorem 8. *For any two subcritical BGW processes $\{Z_1(n)\}$ and $\{Z_2(n)\}$ with offspring means $\mu_2 < \mu_1 < 1$, there exists an n_0 such that $f_n(0) < g_n(0)$ for all $n > n_0$ and therefore T_1 is ultimately dominant, i.e. there exists an n_1 such that $M(n, 0) > M(k, n - k)$ for all $n > n_1$ and $1 \leq k \leq n$.*

Proof. From [10] we know that, as $n \rightarrow \infty$,

$$1 - f_n(0) \sim \mu_1^n L_1(\mu_1^n) \quad \text{and} \quad 1 - g_n(0) \sim \mu_2^n L_2(\mu_2^n),$$

where $L_i(s)$ are functions slowly varying at 0, i.e., for any $c > 0$, $L_i(cs) \sim L_i(s)$ as $s \searrow 0$, $i = 1, 2$. As is well-known (see e.g. Proposition 1.3.6(v) in [4]), if $L(s)$ is slowly varying at 0 then, for any $\varepsilon > 0$, one has $s^\varepsilon < L(s) < s^{-\varepsilon}$ for all small enough $s > 0$. Therefore, for an arbitrarily small ε ,

$$\frac{L_1(s_1)}{L_2(s_2)} > (s_1 s_2)^\varepsilon$$

once s_1 and s_2 are small enough. So, for $0 < \varepsilon < (\ln \mu_2 - \ln \mu_1)/(\ln \mu_2 + \ln \mu_1)$ we obtain

$$\frac{1 - f_n(0)}{1 - g_n(0)} \sim \frac{\mu_1^n L_1(\mu_1^n)}{\mu_2^n L_2(\mu_2^n)} > \left(\frac{\mu_1^{1+\varepsilon}}{\mu_2^{1-\varepsilon}} \right)^n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Hence $f_n(0) < g_n(0)$ for all large enough n . Together with (3) that means that $F(s) < G(s)$ for all large enough s . Therefore T_1 will be ultimately dominant by virtue of Theorem 6. \square

Note that, in the case of critical BGW processes with offspring pgf's f and g , a result analogous to the first part of the above holds true: if the variance of f is lower than that of g , then $f_n(0) > g_n(0)$ for all large enough n (this follows, say, from relation (1) on p. 19 of [2]).

The next results provides an upper bound for the number of dominance changes.

Theorem 9. *Let $\{Z_1(n)\}$ and $\{Z_2(n)\}$ be two subcritical BGW processes with respective offspring pgf's f and g and times to extinction T_1 and T_2 . Then the number of times the difference $f_n(0) - g_n(0)$, $n = 1, 2, \dots$, changes sign and the number of dominance changes for unmixed n -assemblies composed of copies of T_1 and T_2 are no greater than the number of zeroes of $f(\theta) - g(\theta)$, $\theta \in [0, 1]$.*

Proof. Suppose that $f(\theta) \geq g(\theta)$ for $\theta \in [a, b) \subset [0, 1]$ and $f_m(0) \geq g_m(0)$ for some m such that $f_m(0) \in [a, b)$. Then clearly $f_{m+i+1}(0) \geq g_{m+i+1}(0)$ for all $i \geq 0$ such that $f_{m+i}(0) \in [a, b)$. Now supposing that, for some $c \in (b, 1]$, $f(\theta) \leq g(\theta)$ for $\theta \in [b, c)$ and $f_n(0) \leq g_n(0)$ for some n such that $f_n(0) \in [b, c)$, we likewise deduce that $f_{n+j+1}(0) \leq g_{n+j+1}(0)$ for all j such that $f_{n+j}(0) \in [b, c)$. It is easily seen that the above implies that, for any interval $[a, c) \subset [0, 1]$ with at most one zero of $f(\theta) - g(\theta)$, there will be at most one change of sign in the finite subsequence $\{f_n(0) - g_n(0) : f_n(0) \in [a, c), n \in \mathbb{N}\}$.

Suppose that $f(\theta) - g(\theta)$ has $\nu^* < \infty$ zeroes in $[0, 1]$. Then, subdividing $[0, 1]$ into subintervals with one zero in each, it follows that the total number of changes of sign in the sequence $\{f_n(0) - g_n(0) : n \in \mathbb{N}\}$ will not exceed ν^* .

Now let F and G be the respective distribution functions of T_1 and T_2 , i.e. $F(s) = f_{\lfloor s \rfloor}(0)$ and $G(s) = g_{\lfloor s \rfloor}(0)$. Then the number of connected components of $S^+ := \{s : F(s) > G(s)\} = \{s : f_{\lfloor s \rfloor}(0) > g_{\lfloor s \rfloor}(0), s \in \mathbb{R}_+\}$ and $S^- := \{s : F(s) < G(s)\} = \{s : f_{\lfloor s \rfloor}(0) < g_{\lfloor s \rfloor}(0), s \in \mathbb{R}_+\}$ will not exceed ν^* . Applying Theorem 7 completes the proof. \square

Suppose now that the dynamics of the populations of the two species are modeled by subcritical branching processes $\{Z_1(n)\}$ and $\{Z_2(n)\}$ with offspring pgf's f and g , respectively, such that

$$f(\theta) - g(\theta) \quad \text{has at most one zero in } [0, 1]. \quad (4)$$

This is the case, for instance, when both pgf's are quadratic: $f(\theta) = p_0 + p_1\theta + p_2\theta^2$ and $g(\theta) = q_0 + q_1\theta + q_2\theta^2$ with $p_0 + p_1 + p_2 = q_0 + q_1 + q_2 = 1$.

The following result describes the behaviour of $M(k, n-k)$, $0 \leq k \leq n$, depending on agreement of the signs of the differences in respective offspring means and mean times to extinction.

Corollary 10. *Assume that subcritical pgf's f and g satisfy condition (4).*

- (i) *If $(\mu_1 - \mu_2)(\mathbb{E}T_1 - \mathbb{E}T_2) > 0$ then we have a dominant case, with $M(n, 0)$ largest when $\mu_1 > \mu_2$ and $M(0, n)$ largest otherwise.*
- (ii) *If $(\mu_1 - \mu_2)(\mathbb{E}T_1 - \mathbb{E}T_2) < 0$ then we have a non-dominant but ultimately dominant case with exactly one sign change in the sequence $M(n, 0) - M(0, n)$, $n = 1, 2, \dots$*

Proof. (i) Without loss of generality assume that $\mu_1 > \mu_2$. Then, on the one hand, T_1 is ultimately dominant by Theorem 8, so that

$$f_n(0) - g_n(0) < 0 \quad \text{for all large enough } n.$$

On the other hand, due to condition (4), by Theorem 9 the sequence $f_n(0) - g_n(0)$, $n = 1, 2, \dots$, can change sign at most once. Therefore there exists an $n_0 \geq 0$ such that

$$\begin{aligned} f_n(0) - g_n(0) &\geq 0, & n \leq n_0, \\ f_n(0) - g_n(0) &< 0, & n > n_0, \end{aligned}$$

which clearly implies that

$$\sum_{n \geq j} (f_n(0) - g_n(0)) < 0, \quad j > n_0,$$

whereas for $j \leq n_0$ one has

$$\sum_{n \geq j} (f_n(0) - g_n(0)) \leq \sum_{n \geq 0} (f_n(0) - g_n(0)) = \mathbb{E}T_2 - \mathbb{E}T_1 < 0$$

by our assumption. Hence T_1 is dominant by virtue of Theorem 4.

(ii) We again assume that $\mu_1 > \mu_2$, which implies that T_1 is ultimately dominant owing to Theorem 8. However,

$$\mathbb{E}T_1 = M(1, 0) < M(0, 1) = \mathbb{E}T_2$$

by assumption, so there is at least one change of sign in $M(n, 0) - M(0, n)$. Finally, as $f(\theta) - g(\theta)$, $\theta \in [0, 1)$, has at most one zero, Theorem 9 asserts that there cannot be more than one change of sign in the sequence. \square

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